MATH 3060 Tutorial 3

Chan Ki Fung

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1 Review of last tutorial

In tutorial 2, we review the definition of compactness.

Definition 1.1. A subset S of $\mathbb{R}^n$ is called compact if it satisfies any of the following equivalent conditions.

(i) S is closed and bounded.

(ii) Any sequence (x_n) of S has a converging subsequence, i.e. there exists a subsequence (x_{n_k}) and $x \in S$ so that $\lim_k x_{n_k} = x$.

(iii) Any open cover of S has a finite subcover, i.e, If U_α's are open, and $S \subset \cup_\alpha U_\alpha$, then we can cover S using only finitely many $U_\alpha : S \subset U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_n}$.

Remark 1.2. Later in this course when you will learn compactness for more general spaces, you will find the above condition may no longer be equivalent. Condition 3 is the most general one.

Next we discuss about the conditions of Lipschitz.

Definition 1.3. Let $f : [0, 1] \to \mathbb{R}$ be a function, and $x \in [0, 1]$. We have the following conditions:

(i) We say that f is Lipschitz (continuous) at x if there exist $\delta > 0$ and $L > 0$ such that
$$|f(y) - f(x)| \leq L|y - x|$$
whenever $y \in [0, 1]$ and $|y - x| < \delta$.

(ii) We say that f is locally Lipschitz (continuous) at x if there exist $\delta > 0$ and $L > 0$ such that
$$|f(y) - f(z)| \leq L|y - z|$$
whenever $y, z \in [0, 1]$ and $|y - x|, |z - x| < \delta$.

(iii) We say that f is uniformly Lipschitz (continuous) on $[0,1]$ if there exist $\delta > 0$ and $L > 0$ such that

$$|f(y) - f(z)| \leq L|y - z|$$

whenever $x, y \in [0,1]$ and $|x - y| < \delta$.

Remark 1.4.

(i) If f is bounded, then the condition $|y-x| < \delta$ is not necessary in definition (i) and (iii)

(ii) In lecture 4, one can find the definition that f satisfies a Lipschitz condition on $[0,1]$, this is equivalent to the condition in (iii)+ boundedness of f.

It is clear that

uniform Lipschitz $\Longrightarrow$ *locally Lipschitz everywhere* $\Longrightarrow$ *Lipschitz everywhere*

In last tutorial, we show that *locally Lipschitze everywhere* $\Longrightarrow$ *uniform Lipschitz* on compact sets. The proof is similar to the proof that *continuous* $\Longrightarrow$ *uniform continuous* on compact sets.
However, the last condition is not equivalent to the previous one, even on compact sets.

Example 1.5. The function

$$f = \begin{cases} 0, x = 0 \\ x\sin(1/x), x \in (0,1] \end{cases}$$

is Lipschitz at every point. It is a simple exercise to see f is Lipschitz at $x \neq 0$ (using derivatives!). To see f is Lipschitz at $x = 0$, note that $|f(y) - f(0)| = y|\sin(1/y)| \leq |y-0|$. It remains to see that f is not uniformly Lipschitz. In fact, let $x_n = 1/(n\pi + \frac{1}{2}\pi)$, $y_n = 1/(n\pi - \frac{1}{2}\pi)$, we have $|f(x_n) - f(y_n)|/|x_n - y_n| = 2n$.

Finally, we see that if f is differentiable (on $(0,1)$) with bounded derivative, then f is uniformly Lipschitz. On the other hand, if f is uniformly Lipschitz and differentiable, then f' is bounded. However, the function in Example 1.5 also provides an example of a differentiable function, Lipschitz at every point of $[0,1]$, with unbounded derivative.

2 Answers of Last Tutorial's question

(a) If f is differentiable and f' is bounded on $[0,1]$, then f is uniform Lipschitz on $[0,1]$
Ans:True.

(b) If is Lipschitz on $[0,1]$, and f is differentiable, then f' is bounded on $[0,1]$.
Ans: False, a counter example is $x \sin \frac{1}{x}$.

(c) The function $f(x) = x^2$ is uniformly Lipschitz on $[0,1]$.
Ans: True.

(d) There exists no integrable functions f on $[-\pi, \pi]$ so that

$$f \sim \sum_{n=1}^{\infty} \sin nx.$$

True, by Riemann Lebesgue Lemma.

(e) There exists no integrable functions f on $[-\pi, \pi]$ so that

$$f \sim \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cos nx.$$

Ans:True, by Parseval identity.

(f) Let $f_n \to f$ on $[0,1]$ in L^2 sense, then $f_n(x) \to f(x)$ for some $x \in [0,1]$.
Ans: False, we will discuss it in the tutorial.

(g) If $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges uniformly (i.e. the partial sum $s_N = \sum_{n=-N}^{N}$ converges uniformly), then $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$.
Ans: True.

(h) If $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$, then $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges uniformly.
False, if $c_n = 1/n$, then the series diverges for $x = 0$.

(i) Let $c_n = c_n(f)$ for some function f integrable on $[-\pi, \pi]$, then $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges for almost all $x \in [-\pi, \pi]$.
This is true for Riemann integrable functions (but the proof is hard), but incorrect for Lebesgue integrable functions, just forget about this question.

(j) Let f be a 2π periodic continuous, suppose $c_n(f) = 0$ for all n. Then f is the zero function.
Ans: True, using Weierstrass approximation theorem.

Question: Let $0 < \delta < \pi$, and define the 2π periodic function f by

$$f(x) = \begin{cases} 1, & \textit{if } |x| \leq \delta \\ 0, & \textit{if } \delta < |x| \leq |\pi| \end{cases}$$

(a) Compute the Fourier coefficients of f.
Ans:$a_0 = \delta/\pi, a_n = 2 \sin n\delta / n\pi, b_n = 0$.

(b) Show that

$$\sum_{n=1}^{\infty} \frac{\sin n\delta}{n} = \frac{\pi - \delta}{2}.$$

Ans: Evaluate at 0.

(c) Show that

$$\sum_{n=1}^{\infty} \frac{\sin^2 n\delta}{n^2\delta} = \frac{\pi - \delta}{2}.$$

Ans: Use Parseval's identity. (You can check both sides agree when $\delta \to 0$.)

(d) Show that

$$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$$

Ans: Using definition of Riemann sum.

3 Questions for this tutorial

1. True or false
   (a) If f is integrable on $[0,1]$, then f^2 is integrable on $[0,1]$.
   (b) If f^2 is integrable on $[0,1]$, then f is integrable on $[0,1]$.
   (c) If f^2 is integrable on $[0,1]$, then $|f|$ is integrable on $[0,1]$.
   (d) If f is non-negative and continuous on $(0,1]$, and $\int_0^1 f$ exists as an improper integral, then $\int_0^1 f^2$ exists as an improper integral.
   (e) If f is non-negative and continuous on $(0,1]$, and $\int_0^1 f^2$ exists as an improper integral, then $\int_0^1 f$ exists as an improper integral.
2. Let f be a function on $(-\pi, \pi]$, which is integrable on $[a, \pi]$ for any $a \in (-\pi, \pi]$, and that $\lim_{c \to -\pi} \int_c^{\pi} f$ exists, show that Riemann Lebesgue lemma holds.
3. If f is uniformly Lipschitz and 2π periodic, show that $c_n(f) = O(1/n)$.
4. Show that

$$-\log|2\sin\frac{x}{2}| \sim \sum_{n=1}^{\infty} \frac{\cos x}{n}$$

   Hints: $\int_0^{\pi} \log\sin\frac{x}{2} = -\frac{\pi}{2}\log 2$.